MAST30001 2013, Assignment 1
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Instructions: Answer the following questions. Justify all work and give clear, concise explanations, using prose when appropriate. Clarity, neatness and style count.

1. Let X and Y be independent and have exponential distributions with common rate λ (recall this means they each have density $\lambda e^{-\lambda x}$ for $x > 0$) and let $U = X/(X+Y)$ and $V = (X+Y)$.
   (a) What is the marginal density of U?
   (b) What is the joint pdf of (U, V)?
   (c) Are U and V independent?

2. Let N be a positive geometric random variable ($\mathbb{P}(N = j) = p(1-p)^{j-1}$, $j = 1, 2, \ldots$) and let the distribution of T given $N = n$ be gamma with parameter n (given $N = n$, T has density *proportional* to $x^{n-1}e^{-x}$).
   (a) Without computing the density of T, can you find $\mathbb{E}T$?
   (b) What is the joint distribution of (T, N)?
   (c) What is the density of T? Use the density to compute $\mathbb{E}T$ and check that it agrees with your answer in part (a).

3. Let $A_1, A_2, \ldots$ be an iid sequence of random variables uniform on the set $\{2, 3\}$ and for $n \geq 1$ let $B_n = A_{n+1}A_n$. Is $(B_n)_{n\geq 1}$ a Markov chain? If so then analyze its state space (reducibility, periodicity, recurrence, etc) and derive its long term behavior.

4. Let $(X_n)_{n\geq 1}$ be a Markov chain with state space $\{1, \ldots, k\}$ for some $k \geq 1$. Show that if i and j communicate, then the probability that the chain started in state i reaches state j in k steps or fewer is greater than 0.

5. A Markov chain has transition matrix

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix}$$

   Analyze the state space (reducibility, periodicity, recurrence, etc), and discuss the chain's long run behavior.

6. For $i = 0, \ldots, N$ let $0 < p_i = 1 - q_i < 1$ and let the Markov chain (X_n) with state space $\{0, \ldots, N\}$ have transition matrix

$$\begin{pmatrix} q_0 & p_0 & 0 & 0 & 0 & \cdots & 0 \\ q_1 & 0 & p_1 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 0 & p_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & q_{N-2} & 0 & p_{N-2} & 0 \\ 0 & \cdots & 0 & 0 & q_{N-1} & 0 & p_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & q_N & p_N \end{pmatrix}.$$

(a) Analyze the state space of the chain (reducibility, periodicity, recurrence, etc).

(b) If the chain starts at state 0, describe the long run behavior of the chain.

(c) If the chain starts at a uniformly chosen state, describe the long run behavior of the chain.

(d) Given the chain is in state 2 what is the expected number of steps until the chain returns to state 2?

(e) Now assume that $q_N = p_1 = 1$. Analyze the state space of the chain.

7. A possum runs from corner to corner along the top of a square fence. Each time he switches corners, he chooses among the two adjacent corners, choosing the corner in the clockwise direction with probability $0 < p < 1$ and the corner in the counter-clockwise direction with probability $1 - p$. Model the possum's movement among the corners of the fence as a Markov chain, analyze its state space (reducibility, periodicity, recurrence, etc), and discuss its long run behavior.