MAST30001 2013, Assignment 2
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Instructions: Solve the following problems. Justify all work and give clear, concise explanations, using prose when appropriate. Clarity, neatness and style count.

1. You're trying to sell a puppy and you have three friends interested in buying. They each make an offer that you must either immediately take or turn down permanently before seeing the next offer. The first friend tosses a fair coin and offers \$2 if it comes up heads and \$1 if it comes up tails. The offers of friends 2 and 3 depend on the previous offer. If the previous offer is \$1, then the next offer is \$1 with probability 1/3 and \$2 with probability 2/3. If the previous offer is \$2, then the next offer is \$1 with probability 9/10 and \$10 with probability 1/10. If the previous offer is \$10, then there is even chance the next offer is \$1 or \$2. What is the best strategy to maximise the expected selling price and what is this maximum expected price?

2. A two state continuous time Markov chain $(X_t)_{t\geq 0}$ has the following generator with transition rates $\lambda, \mu > 0$:
$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$
   (a) Find the time t transition matrix $P^{(t)}$ with $(P^{(t)})_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$.
   (b) Describe the long run behaviour of the chain.
   (c) Using your answer to part (a) with $\lambda = \mu$, find a simple expression (i.e., not an infinite sum) for the chance that a random variable having the Poisson distribution with mean λ is an even number.

3. A system has N particles each of which at any given time are in one of the two energy states α or β. The particles switch between states α and β according to the following rules. When a particle enters state α, it switches to state β after an exponentially distributed with rate $\mu > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state α. Similarly, when a particle enters state β, it switches to state α after an exponentially distributed with rate $\lambda > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state β.
   (a) Model the number of particles in the energy state α as a continuous time Markov chain and define its generator.
   (b) Describe the long run behaviour of the chain.

(c) If the chain starts with N particles in the α energy state and X_t is the number of α particles at time t, find the mean and variance of X_t as $t \to \infty$. Your answer should be a tidy formula.

4. Suppose a pub sells food and beverages and has three cashiers, A, B, and C. Food orders must be made at cashiers A and B where beverage-only orders are not allowed. Cashier C is exclusively for beverage-only orders. Assume customers ordering food arrive according to a rate 10 Poisson process and customers ordering only beverages arrive independently according to a rate 20 poisson process. Assume also the service times at each of the cashiers are exponentially distributed with rate 60, independent of the arrivals. [Time units are in hours.]

   (a) What is the distribution of the amount of time until the first customer of the night arrives?

   (b) What is the chance that exactly five customers arrive in the first hour after opening?

   (c) What is the chance that exactly five customers arrive in the first hour after opening and exactly two of these are beverage-only customers?

   (d) Given exactly 5 customers have arrived in the first hour after opening, what is the chance exactly two of these are beverage-only customers?

   (e) Given exactly 5 customers have arrived in the first hour after opening, what is the chance that exactly 3 customers arrived within the first 20 minutes of opening?

   (f) What is the chance the next customer that comes in will order food?

   (g) Over many nights the pub is open, about what proportion of time does the beverage-only queue have at least three people?

   (h) What is the average amount of time beverage-only ordering customers have to wait to order a drink?

   (i) Now assume that all cashiers are open to everyone. What is the average amount of time customers have to wait to order? What proportion of customers don't wait at all?