Exercises 1, Stat 206B/Math 223B, Spring 2011

1. Find a characterizing operator for the normal distribution with mean μ and variance σ^2. You should check your answer by deriving the mean and variance of the distribution using your operator. Recall that the characterizing operator for the standard normal distribution is $\mathcal{A}$ defined by

$$(\mathcal{A}f)(x) = f'(x) - xf(x).$$

2. Let $f_x(w)$ be the bounded function that satisfies

$$f_x'(w) - wf_x(w) = \mathrm{I}[w \leqslant x] - \Phi(x),$$

where $\Phi(x)$ is the standard normal distribution function. Using that

$$1 - \Phi(w) \leqslant \min\left\{\frac{1}{2}, \frac{1}{w\sqrt{2\pi}}\right\} e^{-w^2/2}, \ w \geqslant 0,$$

show $\|f'\| \leqslant 2$.

3. Show the Mills ratio bound for $w > 0$:

$$\frac{we^{-w^2/2}}{(1+w^2)\sqrt{2\pi}} \leqslant 1 - \Phi(w) \leqslant \frac{e^{-w^2/2}}{w\sqrt{2\pi}}.$$

4. Let X and Y integer valued random variable and recall the definition of total variation distance in this case:

$$d_{\mathrm{TV}}(X,Y) = \sup_{A \subseteq \mathbb{Z}} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|.$$

Show that

$$d_{\mathrm{TV}}(X,Y) = \frac{1}{2}\sum_{j \in \mathbb{Z}} |\mathbb{P}(X = j) - \mathbb{P}(Y = j)|.$$

5. Let $X_1, \ldots, X_n$ i.i.d. with $\mathbb{P}(X_i = 1) = p = 1 - \mathbb{P}(X_i = 0)$, and for some k with $1 \leqslant k \leqslant n$, define

$$Y_j = \prod_{i=j}^{j+k-1} X_i, \ j = 1, \ldots, n,$$

where the bounds of the product are defined modularly (e.g. $Y_n = X_n X_1 \cdots X_{k-1}$). Let

$$W = \frac{\sum_{j=1}^{n} Y_j - \mathbb{E}\left[\sum_{j=1}^{n} Y_j\right]}{\sqrt{\mathrm{Var}\left(\sum_{j=1}^{n} Y_j\right)}}$$

and let d_{W} denote the Wasserstein distance.

(a) Using the method of dependency graphs, find a bound on $d_{\mathrm{W}}(W, Z)$, where Z has the standard normal distribution. It may help you to first answer the question for small values of $k = 1, 2, 3$ and then try to generalize.

(b) Does your bound in the previous item go to zero for k fixed? At what rate? How fast can k grow with n so that your bound still goes to zero?