Exercises 2, Stat 206B/Math 223B, Spring 2011

1. Let $X_0, X_1, \dots$ be a reversible time homogeneous Markov chain in stationary on a discrete state space Ω, and let W be a random variable on Ω. Show that $(W(X_0), W(X_1))$ is an exchangeable pair.

2. Show that for X a random variable and two sigma fields $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we have
$$\mathrm{Var}(\mathbb{E}[X|\mathcal{F}_1]) \leqslant \mathrm{Var}(\mathbb{E}[X|\mathcal{F}_2]).$$
Verify this inequality by computing the relevant quantities for $X = \sum_{i=1}^{k} X_i$, where $1 \leqslant k \leqslant n$, $X_1, X_2, \dots, X_n$ are i.i.d. Bernoulli random variables, $\mathcal{F}_1 = \sigma(\sum_{i=1}^{n} X_i)$, and $\mathcal{F}_2 = \sigma(\{X_1, X_2, \dots, X_n\})$.

3. Let (W, W') an exchangeable pair of random variables such that $\mathbb{E}[W] = 0$, $\mathbb{E}[W^2] = 1$, and for some sigma field $\mathcal{F} \supseteq \sigma(W)$ we have
$$\mathbb{E}[W' - W|\mathcal{F}] = -aW + R,$$
for some $0 < a \leqslant 1$ and R a random variable (such a decomposition will always exist). For Z a standard normal random variable show that
$$d_{\mathrm{W}}(W, Z) \leqslant \sqrt{\frac{2}{\pi}} \frac{\sqrt{\mathrm{Var}\left(\mathbb{E}\left[(W' - W)^2|\mathcal{F}\right]\right)}}{2a} + \frac{\mathbb{E}\left|W' - W\right|^3}{3a} + \frac{\sqrt{\mathrm{Var}(R)}}{a}.$$

4. Recall that an Erdős-Rényi graph on n vertices with edge probability p is a random graph on n vertices where each of the $\binom{n}{2}$ edges appears independently with probability p.

   (a) Use Problem 3 to obtain an upper bound on the Wasserstein distance between a standard normal distribution and the number of isolated vertices in an Erdős-Rényi random graph on n vertices with edge probability p (properly normalized).

   (b) It is known that the number of isolated vertices in an Erdős-Rényi random graph satisfies a central limit theorem as n goes to infinity provided that
   $$\lim_{n\to\infty} n^2 p = \lim_{n\to\infty} (\log(n) - np) = \infty.$$
   Does your bound in part (a) go to zero if $np \to c \in (0, \infty)$? Can you find the regions of this regime where your bound in part (a) goes to zero?

5. Let $X_1, \dots, X_n$ be i.i.d. non-negative random variables and let with $\mathbb{E}[X_1] = \mu$, $\mathrm{Var}(X_1) = \sigma^2$, and $\mathbb{E}[X_1^3] = \gamma$. Use the size bias transform to find an upper bound of
$$d_{\mathrm{W}}\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}, Z\right),$$
where Z is a standard normal random variable. Your answer should be in terms of μ, σ, γ, and n.