MAST30001 2013, Recommended Problems, Chapter 3
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Instructions: Answer the following questions. Clearly show all work and give clear and concise explanations, using prose when appropriate. Clarity, neatness and style count.

1. For a discrete time Markov chain on a discrete state space, say $\Omega = \{1, 2, \ldots\}$, show that for $k, j, x_1, \ldots x_{n-1} \in \Omega$,

$$\mathbb{P}(X_{n+m} = k | X_n = j, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1) = \mathbb{P}(X_{n+m} = k | X_n = j)$$

   and explain what this means in words.

2. A Markov chain with transition matrix

$$\begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{pmatrix}$$

   starts with initial distribution uniform on the states $1, 2, 3, 4$. For each $i = 1, 2, 3, 4$, and $n \geq 0$, what is the chance the chain is in state i at step n?

3. An urn contains blue balls and red balls. At discrete time steps, a ball is chosen uniformly at random from the urn and replaced along with another ball of the same color so the total number of balls in the urn increases by one. Let (b_0, r_0) be the number of blue and red balls the urn initially contains and (B_n, R_n) be the number of blue and red balls in the urn after n steps of this process.

   (a) What is $B_n + R_n$ in terms of n, b_0, r_0?

   (b) Is (B_n, R_n) a Markov chain? What are the transition probabilities?

   (c) Analyze the state space of the chain (communication classes, essentiality, reducibility, recurrence, periodicity, etc).

   (d) Is B_n a Markov chain?

   (e) If $b_0 = r_0 = 1$, what is the distribution of B_2?

   (f) If $b_0 = r_0 = 1$, show that B_{100} is distributed uniformly on $\{1, \ldots, 101\}$.

4. Two containers labelled α and ω have $2k$ balls distributed between them. At discrete time steps a ball is uniformly chosen from all $2k$ balls and it is moved from the container it is in to the other container. Let X_n be the number of balls in container α after the nth time step. This is a simple model for molecules diffusing through a membrane.

(a) Is X_n a Markov chain? What are the transition probabilities?

(b) Analyze the state space of the chain (communication classes, essentiality, reducibility, recurrence, periodicity, etc).

(c) If X_0 has a binomial distribution with parameters $(2k, 1/2)$ (meaning that X_0 balls are initially put into container α and $2k - X_0$ are put into container ω), what is the distribution of X_{10}?