MAST30001 2013, Recommended Problems, Chapter 3
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Instructions: Answer the following questions. Clearly show all work and give clear and concise explanations, using prose when appropriate. Clarity, neatness and style count.

1. For a Markov chain with state space S and n step transition probabilities $p_{ij}^{(n)}$, let $f_{j,k}$ be the chance that the chain started in state j ever reaches k. If
$$\sum_{n=0}^{\infty} p_{jj}^{(n)} + \sum_{k \neq j} f_{j,k} \sum_{n=0}^{\infty} p_{kk}^{(n)}$$
is finite, then show that it equals
$$\sum_{k \in S} \sum_{n=1}^{\infty} p_{jk}^{(n)}.$$
It may help to use the probabilities, say $f_{j,k}(l)$, of the chance that the chain started from state j reaches state k for the first time in exactly l steps.

2. Refer to problem 8, Section 3.7 of the textbook (Elements of Stochastic Modelling by K. Borovkov), assume that in Corner 2 there is a bigger spider ready to eat the little spider and in Corner 3, there is a hole leading to the outside through which the spider can escape.
   (a) If the spider starts in Corner 1, what is the chance the spider will escape before being eaten?
   (b) If the spider starts in Corner 1, what is the expected number of steps before the spider exits (either the box or this world)?
   (c) If initially the spider is dropped in the middle of the box and it chooses a corner uniformly, what is the chance the spider will escape before being eaten?
   (d) If initially the spider is dropped in the middle of the box and it chooses a corner uniformly, what is the expected number of steps before the spider exits (either the box or this world)?

3. A simplified model for the spread of a contagion in a small population of size 5 is as follows. At each discrete time unit, two individuals in the population are chosen uniformly at random to meet. If one of these persons is healthy and the other has the contagion, then with probability 1/4 the healthy person becomes sick. Otherwise the system stays the same. If initially one person has the disease, what is the average amount of time before everyone in the population has the disease? What about if the population is of size N?