MAST30001 2013, Recommended Problems, Chapter 5
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Instructions: Answer the following questions. Justify all work and give clear, concise explanations, using prose when appropriate. Clarity, neatness and style count.

1. (Discrete version of Poisson Process) Let the discrete time Markov chain $(X_n)_{n\geq 0}$ on $\{0, 1, \ldots\}$ have transition probabilities $p_{ii+1} = 1 - p_{ii} = p$ and assume $X_0 = 0$.

   (a) Draw a picture of a typical trajectory of this process.

   (b) Show that $(X_n)_{n\geq 0}$ has the independent increments property: for $0 \leq i < j \leq k < l$, the variables
   $$(X_l - X_k, X_j - X_i)$$
   are independent.

   (c) Show that X_n has the binomial distribution with parameters n and p.

   (d) Show that for $m < n$, $X_n - X_m$ has the binomial distribution with parameters $n - m$ and p.

   (e) Show that the number of steps between "jumps" (times when the chain changes states) has the geometric distribution with parameter p (and started from 1).

   (f) Show that given $X_n = 1$, the step number of the first jump is uniform on $\{1, \ldots, n\}$.

2. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and for each $t \geq 0$, let $X_t = N_{t/\lambda}$. Show that $(X_t)_{t\geq 0}$ is a Poisson process with rate 1.

3. In a Poisson process with rate 1, what is the joint density of the times of the first and second jumps? What is the joint density of the times of the ith and jth jump for $i < j$? Can you interpret these formulas similar to our discussion in lecture deriving the joint densities of order statistics?

4. Let $U_{(1)}, \ldots, U_{(n)}$ be order statistics of independent variables, uniform on the interval $(0, 1)$. For $0 < x < y < 1$ what is

   (a) $\mathbb{P}(U_{(1)} > x, U_{(n)} < y)$,

   (b) $\mathbb{P}(U_{(1)} < x, U_{(n)} < y)$,

   (c) $\mathbb{P}(U_{(k)} < x, U_{(k+1)} > y)$,

   (d) $\mathbb{P}(U_{(k)} < x, U_{(k+2)} > y)$?

5. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ.

   (a) What is $\mathbb{P}(N_1 = 2|N_2 = 10)$?

6. From Assignment 1: If N is geometric with parameter p ($\mathbb{P}(N = j) = (1-p)^{j-1}$, $j = 1, 2, \ldots$) and given $N = n$, T has density

$$f_{T|N}(t|n) = \frac{t^{n-1}e^{-t}}{(n-1)!},$$

   what is the density of T? Another question: If S is exponential with rate λ and given $S = s$, M is Poisson with mean s, then what is the distribution of M? A third question: If K is Poisson with mean μ and given $K = k$, J is binomial with parameters k and p, then what is the distribution of J? Can you explain (or even derive) the answers to these three questions through superposition and thinning of Poisson processes?

7. In a certain electrical system, spikes of current arrive according to a Poisson process with rate λ and the size of each spike has an exponential distribution with rate μ amperes, independent of the time of arrival of the spike and the size of the other spikes. A fuse in the system can tolerate up to a total of a amperes cumulatively over time before failure. What is the expected amount of time a new fuse will last before failure? *Hint: you may want to use the formula for a non-negative random variable T, $\mathbb{E}T = \int_0^\infty \mathbb{P}(T > t)dt$.*

8. Let S be a rate one exponential variable and given $S = s$, let X_t be a Poisson process with rate s. Find $\mathbb{P}(X_t = j)$ for $t > 0$ and $j = 0, 1, \ldots$