MAST30001 2013, Recommended Problems, Chapter 6
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Instructions: Answer the following questions. Justify all work and give clear, concise explanations, using prose when appropriate. Clarity, neatness and style count.

1. (DTMCs embedded in CTMCs) Let P be a one step transition matrix for a discrete time Markov chain on $0, 1 \ldots$ such that $p_{ii} = 0$ for all i and define the following continuous time (homogeneous) Markov chain. Given $X_0 = i$ the chain waits an exponential with rate λ_i time and then jumps to state j with probability p_{ij} (the state jumped to is independent of the time of the jump). What is the generator of this continuous time Markov chain? What is the relationship between the classification of states of the discrete and continuous time chains? What is the relationship between the long run behaviour of the discrete and continuous time chains?

2. (CTMCs as limits of DTMCs) Let P be a one step transition matrix for a discrete time Markov chain on $0, 1 \ldots$ such that $p_{ii} = 0$ for all i. Also let $0 < \lambda_0, \lambda_1, \ldots$ be such that $\max_{i \geq 0} \lambda_i < N$, with N an integer. Define the discrete time Markov chain $Y_0, Y_1, \ldots$ by
$$\mathbb{P}(Y_{n+1}^{(N)} = i | Y_n^{(N)} = i) = \left(1 - \frac{\lambda_i}{N}\right),$$
and for $i \neq j$
$$\mathbb{P}(Y_{n+1}^{(N)} = j | Y_n^{(N)} = i) = \frac{\lambda_i}{N} p_{ij}.$$
We can embed $Y^{(N)}$ into the lattice $\{0, 1/N, 2/N \ldots\}$ to make a continuous time process by defining
$$X_t^{(N)} = Y_{\lfloor NT \rfloor}^{(N)},$$
where $\lfloor a \rfloor$ is the greatest integer not bigger than a.

   (a) What does a typical trajectory of $X^{(N)}$ look like? Does it have jumps? At what time? How do jumps correspond to $Y^{(N)}$?

   (b) Given $X_0^{(N)} = i$, what is the distribution of the random time
$$T^{(N)}(i) = \min\{t \geq 0 : X_t^{(N)} \neq i\}$$

   (c) As $N \to \infty$, to what distribution does that of the previous item converge?

   (d) Based on the previous two items and comparing to the previous problem, do you think that $X^{(N)}$ converges as $N \to \infty$ to a continuous time Markov chain (not worrying about what exactly convergence means)? What is its generator?

3. The following continuous time Markov chain is used to model population growth without death. The basic assumption of the model is that every member of the population gives birth to a new member with rate λ (that is, at times with distribution exponential with rate λ), independently of the other members of the population. Let X_t be the size of the population at time t.

   (a) What is $\mathbb{P}(X_t = n | X_0 = 1)$?

   (b) If U is uniform on the interval $(0, 1)$, independent of X_t, find the distribution of $X_U | X_0 = 1$.

4. If X_t is a pure death chain started from state N with death rates $\mu_1, \ldots, \mu_N$ and T is an independent exponential variable with rate λ, show that

$$\mathbb{P}(X_T = 0) = \prod_{i=1}^{N} \frac{\mu_i}{\mu_i + \lambda}.$$

5. If $(X_t^{(1)})_{t \geq 0}, \ldots, (X_t^{(k)})_{t \geq 0}$ are iid continuous time Markov chains on $\{0, 1\}$ each having generator

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$

then what is the generator for the chain determined by $Y_t = \sum_{i=1}^{k} X_t^{(i)}$?